

NOTE ON THE CALCULATION OF THE PROBABILITIES OF LIFE AT HIGH AGES.

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Note on the Calculation of the Probabilities of Life at High Ages.

By JOHN BROWNLEE and R. M. MORISON.

THE calculation of life-tables at the high ages is in general unsatisfactory. Neither are the available data above reproach, nor are the methods used in dealing with them fully justified. The need for the present investigation became evident in the course of some work at certain of the facts of immunity, which could not be satisfactorily understood without a fuller knowledge of the sort of error which occurs in these tables; it makes no pretence to be a complete investigation.

The data are of two kinds—the age-numbers of the population given at the census periods, and the age deaths obtained from the ages of death as certified by the friends. This certification has varied in accuracy, and the direction of error has changed with the times. Before 1880 the tendency on the part of the friends was to exaggerate the ages of those whose deaths they registered, while since that time it has been to understate them. The inaccuracy of the past probably arose from a sort of vanity, that of the present is due to the fact that wrong ages have already been given for insurance purposes. It is likely that in the immediate future old age pensions will cause a swing of the error in the previous direction. The tables obtained from the census numbers and from the ages at death are easily compared, for at the higher ages the population has remained fairly constant. In order that this constancy may be maintained, the people who die above a certain age in any year must be balanced by those who cross that age. Therefore, to get the population at any age from the death returns, we have only to sum all the deaths above that age. The number obtained by interpolation from the census figures is compared with this, and the comparison gives a rough test of correspondence. The result for females in Scotland is shown in Table 1.

TABLE 1.—*Showing the number of females in Scotland living at high ages at each Census from 1871 to 1901, with the corresponding average yearly deaths above those ages for each decade.*

Year.	Persons being of age			Decade.	Average yearly number of deaths above		
	70.	80.	90.		70.	80.	90.
1871	7023·9	2584·9	356·5	1861-71	6495·9	2862·1	469·1
'81	7425·3	2891·9	345·6	'71-81	7348·8	2944·6	450·7
'91 ...	8132·9	3234·6	426·1	'81-91	7310·5	3060·1	426·7
1901	8737·3	3412·7	422·4	'91-1901	8142·0	3238·8	460·1

It is evident that the two sets of figures roughly correspond, and, therefore, the same kind of error, whatever its nature, probably

affects both. The census numbers, however, are in defect at the highest ages. Neither set gives directly any method by which the statistics may be tested. There are, however, two indirect tests, both applicable at ages above 50 years.

1. The change of death-rate with age may be taken as comprehended in a formula and a life-table constructed on the hypothesis of the death-rate alone.

2. The chance of life at any age may be obtained from the figures of the different censuses, for those who are 50 years old at one census should be 60 at the next, and 70 at the next again. Emigration will introduce an error; but emigration does not take place to any extent above the age of 50, and becomes a factor of less and less importance as age advances. The result of the emigration error will be to cause a slight underestimation of the chance of life, decreasing with age. This test is not of much use for local tables, but it is of great value when whole countries are considered. It affords, in particular, a ready method of comparing the mortalities of different countries when the death numbers are not available.

I. *Law of death-rate at high ages.*

Makeham's hypothesis ceases to apply beyond the age of 85, and so the formula derived from it is unavailable for our purpose, but the earlier formula of Gompertz appears to be eminently suitable. The supposition underlying it is that the death-rate increases in a geometrical progression with age; and observation of the tables shows that this is almost exactly true both for males and for females above the age of 50, the death-rate being approximately doubled with increase of ten years in life.

TABLE 2.

Age.	Males.		Females.	
	Death-rates per 1,000.		Death-rates per 1,000.	
	Actual.	Theoretical.	Actual.	Theoretical.
30—35 ...	8·2	—	8·4	—
35—40 ...	10·4	—	9·7	—
40—45 ...	12·4	11·1	11·9	—
45—50 ...	16·2	15·7	12·6	12·2
50—55 ...	22·0	22·2	17·7	17·4
55—60 ...	31·2	31·4	25·2	24·9
60—65 ...	40·4	44·4	33·1	35·6
65—70	58·7	62·9	49·4	50·8
70—75	81·3	88·9	69·6	72·7
75—80 ...	121·6	125·87	106·5	103·8
80—85	198·9	178·1	146·5	148·3
85—90 ...	255·9	252·0	236·6	211·9
90—95 ...	383·9	356·9	318·1	302·8
95—100	421·1	504·4	396·4	432·8
100—105 ...	530·0	—	400·0	—

Table 2 shows the correspondence between the figures obtained for Scotland from the death-tables and those obtained theoretically, on the assumption that such a law is obeyed. The figures represent the average death-rates over five-yearly age periods for the decade 1891-1901. The theoretical numbers were calculated from the others, between the ages of 40 and 95, by use of the method of least squares as applied to the logarithms.

With the females the correspondence is almost exact above the age of 45, with the males less so. Above 95 the theoretical numbers are larger. The figures for England are equally suggestive.

TABLE 3.—*Showing the death-rates per 1,000 of males and females for three five-yearly periods for England at the high ages.*

		55—65.	65—75.	75—85.	85—
Males	1886-90	35.2	72.1	147.9	313.8
	'91-95.....	35.9	72.5	149.3	291.0
	'96-1900....	34.1	68.3	142.9	282.6
Females {	1886-90.....	28.8	61.7	132.3	276.2
	'91-95.....	29.5	63.1	134.4	264.2
	'96-1900 ..	27.4	58.4	126.8	258.5

Annual Report of Registrar-General of England, 1904, p. cxxii.

The almost absolute correspondence for males and females in England and for females in Scotland makes the increase of the death rate in a geometrical progression with age a probable law, and offers an approximation of at least as great accuracy as the methods at present in use for forming life tables at high ages. The males in Scotland do not approach the law so closely. From the age of 60 to the age of 80 there is a considerable defect in the deaths. Why this should be so is not clear.

If a law like this exist, and it is not *a priori* improbable (in the light of modern chemistry it means that the disintegration of some substance necessary to life follows the law of the monomolecular reaction), a formula is easily devised from this source for the population at any age. The following proof is really that of Gompertz, expressed in modern notation.

The death rate per unit time t is evidently $b e^{\alpha t}$ where b is a constant calculated from the data for the period of the origin of the curve and α is a constant independent of the origin.

If x be the number of persons living at a given time t , the number dying at that instant is $x b e^{\alpha t} dt$, which is equal to the decrease out of x , or $-dx = x b e^{\alpha t} dt$. Whence $x = C e^{-\frac{b}{\alpha} e^{\alpha t}}$, C being the constant of integration.

To form a table for the high ages then, all that is necessary is to find the best values of b and α from the death-rate data, and thence to calculate the survivors at any time by the above formula. C is determined from the initial value of x . This formula throws some light on the theory of the life-tables at present used. In them

the high age-values are calculated by logarithmic interpolation. This assumes that the curve most suitable for the graduation of the life-tables is of the form $x = e^{a + bt + ct^2 + \dots}$. The present reasoning justifies this form as suitable. But it must be noticed that the age at which the formula becomes applicable is 50, and that therefore, for extrapolation for high-age calculations, no data earlier than this should be used. Probably because of neglect of this rule life-tables generally give fewer persons living at high ages than the formula. A comparison is shown in the following table. The theoretical values have been obtained by calculating C , b , and a for the first three values of the life-table, and thence calculating the higher-age values.

TABLE 4.

Age.	English life-table.*		Age.	Scottish life-table.†	
	1891-1901 (Females).			1891-1901 (Females).	
	Actual.	Theoretical.		Actual.	Theoretical.
55.....	535470	535470	55.....	516470	516470
65.....	402920	402920	65.....	383416	383416
75.....	219500	219500	75.....	209000	209000
85.....	59352	60048	85.....	55811	60724
95.....	4281	3757.4	95.....	2338	4898
105.....	—	10.137	105.....	4	29

* Dr. Hayward, *Journal of the Royal Statistical Society*, vol. xlv, part ii.

† Dr. Adam, *Journal of the Royal Statistical Society*, vol. xlvii, part iii.

Dr. Hayward's values approach the formula much more closely than those of Dr. Adam. The theoretical values at 95 and upwards are much in excess of the table values. Though the theoretical death rates are in excess of those observed at the highest ages, more persons are found living at those ages than is generally thought possible. When it is remembered that in ten years two persons, one aged 103 and one aged 108, with apparently authentic ages, died in Guernsey, an island having a population of only 40,000 inhabitants, the numbers given at age 105 in Scotland do not seem excessive, twenty-nine persons in Dr. Adam's life tables corresponding to a little over one out of 2,000,000 female inhabitants.

II.

The other check on life table calculations remains to be considered. It is evident that a person who is 50 years old at one census will be 60 years old at the succeeding census ten years later. If no emigration takes place, then the number of persons at 60 years old divided by the number at 50 years old gives the chance of living ten years.

It is obvious that the possible error of this method is in defect of the true value of the ten years' life chance rather than in excess,

since the emigration of people who are 50 at one census gives fewer people of 60 at the next. As a check we have the population estimated from the deaths recorded by summing them from any point to the limit of the highest ages. This is more and more in error the earlier the age chosen as a starting point, but at higher ages is more nearly correct. As another check we may use the formula already discussed. The results for females in Scotland for the several decades since 1871 are given for comparison in Table 5.

TABLE 5.—*Showing the chance of living ten years at various ages as found by the various methods.*

Method.	Chance of living ten years at age				
	50.	60.	70.	80.	90.
Comparison of the census					
Of 1871 and 1881	·7812	·6573	·4117	·1337	—
" '81 " '91	·7814	·6631	·4356	·1423	—
" '91 " 1901	·7871	·6605	·4496	·1305	—
Do. Logarithmic interpolation	—	—	·4116	·1330	·0217
Summation of deaths in decades (from the register)					
1871-81.....	—	·7280	·4007	·1531	·0443
'81-91.....	—	·6497	·4186	·1394	·0441
'91-1901.....	—	·6405	·3978	·1421	·0280
Dr. Adam's life table	·8111	·6514	·4126	·1288	·0094
Formula method.....	·8115	·6536	·4206	·1713	·0275

It may be seen from this that the new census method and the death register give essentially the same results, and that Dr. Adam's table gives results much in defect at 80 years and very much in defect at 90 years. His method, based on extrapolation for values obtained prior to the age 85, evidently goes too far back, and is therefore untrustworthy. The formula method gives a very high value at 80, but one in essential agreement with those obtained from the census figures and from the deaths alone at the age of 90. As the formula from which these results were calculated was obtained from Dr. Adam's figures at the years 55, 65 and 75, it is obvious that essential agreement with Dr. Adam's results was to be expected at 50, 60 and 70. As a further comparison of the results of the new census method with those of the ordinary methods of forming a life table, the following figures for England may be given:—

TABLE 6.—*Showing the chance of living ten years at certain ages.*

Years.	Chance of living ten years at age of			
	English life-tables.		Census method.	
	60.	70.	60.	70.
1871-1881....	·6556	·3929	·6580	·3933
'81- '91....	·6537	·3827	·6732	·4023

Registrar-General's Supplement, part i. Decennium, 1881-1891.

The correspondence is much closer in the first decade than in the second. At both age periods in the latter the life table is in some defect.

This method of comparing the probability of living at high ages affords an opportunity of examining the statistics of various countries. In these a like method has been adopted, the figures being obtained from Dr. Jacques Bertillon's "*Recensements de la Population exécuté dans les divers pays de l'Europe.*" Some difficulty has been experienced from the changes of boundary and from the absence of census returns at certain dates in various European States during the last forty years, but the following results have been calculated:—

TABLE 7.—*Showing the chance of females of living ten years at various ages in certain countries in Europe.*

Country.	Period.	Age.		
		50.	60.	70.
Italy	1871-81	·7953	·5723	·3284
Portugal	'81-91	·8051	·6429	·4235
France.....	'81-91	·8090	·6326	·3701
Belgium	'80-90	·8311	·6983	·4051
Holland	'81-91	·8403	·6988	·4094
Bavaria	'80-90	·7871	·6202	·2736
Hungary	'80-90	·7508	·5455	·3024
Finland	'80-90	·8339	·6698	·3646
Sweden	'80-90	·8689	·7361	·4703

These results are too few to make the basis of many deductions, but the northern and southern races show distinctly more longevity than the central European races, as exemplified by Bavaria and Hungary.

The results of the investigations are: (1) At the highest ages the death rate numbers are in excess of those got from the census tables. (2) The formula method, though it gives a higher death rate, gives still more people living at the highest ages. (3) The error of tables as now constructed is probably principally in the direction of giving too few people at the highest ages.